Minimum Inaccuracy for Traversal-Time

J. Oppenheim[(a)],[*] B. Reznik[(b)],[†] and W. G. Unruh[(a)‡]

(a) Department of Physics, University of British Columbia, 6224 Agricultural Rd. Vancouver, B.C., Canada V6T1Z1

(b) Theoretical Division, T-6, MS B288, Los Alamos National Laboratory, Los Alamos, NM, 87545

Abstract

Using various model clocks it has been shown that the time-of-arrival cannot be measured more accurately than $\delta T_A > 1/E_p$ where E_p is the kinetic energy of a free particle. However, this result has never been proved. In this paper, we show that a violation of the above limitation for the traversal-time implies a violation of the Heisenberg uncertainty relation.

[*] *jono@physics.ubc.ca*

[†] *reznik@t6-serv.lanl.gov*

[‡] *unruh@physics.ubc.ca*

I. INTRODUCTION

In [1], we considered various clock models for measuring the time it takes for a free particle to arrive to a given location x_A. Because the energy of the clock increases with its precision, we argued that the accuracy of a time-of-arrival detector cannot be greater than $1/E_p$, where E_p is the kinetic energy of the particle. Measurements of traversal-time [2] are analogous to that of time-of-arrival. One tries to measure how long it takes a particle to travel between two fixed locations x_1 and x_2. Although no proof has yet been found for the restriction on time-of-arrival accuracy, in this paper we are able to show that a necessary minimum inaccuracy on traversal-time measurements is given by

$$\delta T_F > 1/E_p. \tag{1}$$

We do this by arguing that a traversal-time measurement is also a simultaneous measurement of position and momentum, and that (1) is required in order to preserve the Heisenberg uncertainty relationship. Note however that (1) is not analogous to the Heisenberg Energy-time uncertainty relationship. It reflects the inherent inaccuracy of every individual measurement, while the Heisenberg uncertainty relationships refer to well-defined and perfectly accurate measurements made on ensembles.

The article proceeds as follows. In section II we motivate the notion that traversal-time is a measurement of momentum by looking at measuring the traversal-distance. In section III we discuss a physical model for measuring the traversal-time, and show the relation between (1) and the uncertainty principle. The main result of this paper is given in Section IV, where we provide a model independent derivation of (1), as well as a qualitative proof.

II. MEASURING MOMENTUM THROUGH TRAVERSAL-DISTANCE

The measurement of traversal-distance may be considered the space-time "dual" of the measurement of traversal-time: instead of fixing x_1 and x_2 and measuring $t_F = t_2 - t_1$, one

fixes t_1 and t_2 and measures $x_F = x_2 - x_1$. It is instructive to examine first this simpler case of traversal-distance and point out the similarities and the differences.

Unlike the case of traversal-time, a measurement of traversal-distance can be described by the standard von Neumann interaction. For a free particle the Hamiltonian is

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \mathbf{Q}\mathbf{x}\Big[\delta(t - t_1) - \delta(t - t_2)\Big] \tag{2}$$

where $\mathbf{Q}$ is the coordinate conjugate to the pointer variable $\mathbf{P}$. The change in $\mathbf{P}$ yields the traversal-distance:

$$\mathbf{P}(t > t_2) - \mathbf{P}_0 = \mathbf{x}(t_2) - \mathbf{x}(t_1) = \mathbf{x}_F. \tag{3}$$

However the measurement of the traversal-distance provides additional information: it also determines the momentum $\mathbf{p}$ of the particle *during* the time interval $t_1 < t < t_2$. From the equations of motion we get:

$$\mathbf{p}(t) = \begin{cases} \mathbf{p}_o, & t < t_1 \text{ or } t > t_2 \\ \mathbf{p}_o - \mathbf{Q}, & t_1 < t < t_2 \end{cases} \tag{4}$$

and

$$\mathbf{x}(t) = \begin{cases} x_0 + \frac{\mathbf{p}_o}{m} t, & t \le t_1 \\ x_0 + \frac{\mathbf{p}_0}{m} t_1 + \frac{\mathbf{p}_0 - \mathbf{Q}}{m}(t - t_1), & t_1 \le t \le t_2 \end{cases} \tag{5}$$

and therefore,

$$m\frac{\mathbf{P}(t > t_2) - \mathbf{P}_0}{t_2 - t_1} = \mathbf{p}_0 - \mathbf{Q} = \mathbf{p}(t_1 \le t \le t_2). \tag{6}$$

Thus, one can determine simultaneously and to arbitrary accuracy the traversal-distance and the momentum in intermediate times. This, of course, does not contradict the uncertainty relations, because $\mathbf{p}$ commutes with $\mathbf{x}_F$, and $\mathbf{x}$ remains completely uncertain. Similarly, in the case of the traversal-time we shall see that the measurement determines also the intermediate time momentum, however unlike the present case, since the particle has to be in the interval $x_2 - x_1$ during the traversal, it is also a measurement of the location. This indicates that, in the latter case, in order not to violate the uncertainty principle, the accuracy with which T_F or p may be measured must be limited.

III. MEASURING TRAVERSAL-TIME

In quantum mechanics, classical observables such as position, momentum and energy are represented by an operator $\mathbf{A}$ whose eigenvalues give the possible outcomes of a measurement. However, some classical observables, such as time [3] and time-of-arrival [1] [4] cannot be represented by operators. For example, for time-of-arrival, one can use the correspondence principle to find the operator (up to ordering difficulties)

$$\mathbf{T_A} = m(\frac{1}{\mathbf{p}}\mathbf{x} + \mathbf{x}\frac{1}{\mathbf{p}}). \tag{7}$$

However it turns out that due to the singularity at $p = 0$, the eigenstates of this operator are not orthogonal and therefore $\mathbf{T_A}$ is not Hermitian. One could regularize this operator in some way [5] however the resulting operator is unphysical. Measuring this operator is not equivalent to physically measuring the time-of-arrival [1].

For traversal-time the situation is similar. The classical equations of motion suggest that a traversal-time operator might be given by

$$\mathbf{T_F} = \frac{mL}{\mathbf{p}}, \tag{8}$$

where $L = x_2 - x_1$. Like the time-of-arrival operator, this operator is undefined at $p = 0$, and again different outcomes are found in a direct measurement of T_F and a measurement of a regularized $\mathbf{T_F}$. One can measure the momentum at any time, so there is no reason to believe that the particle actually travelled between the two points in the time t_F. A measurement of $1/\mathbf{p}$ will result in the particle's position being spread over all space, so there is no finite amount of time one could wait before being certain that the particle went between the two fixed points. For example, after the measurement of $1/\mathbf{p}$, the potential between x_1 and x_2 might change. General traversal-time operators would require that one knows the Hamiltonian not only in the past, but also in the future. If one measures the traversal-time operator above, then one has to have faith that the Hamiltonian will not change after the time of the measurement t_o to $t \to \infty$.

It is also commonly accepted that the dwell time operator [6], given by

$$\tau_{\mathbf{D}} = \int_0^\infty dt \mathbf{\Pi}(t) \tag{9}$$

where

$$\mathbf{\Pi}(0) = \int_{x_1}^{x_2} |x\rangle\langle x| \tag{10}$$

can be used to compute the traversal time[1]. Such a quantity however, cannot be measured, since the operator $\mathbf{\Pi}(t)$ does not commute with itself at different times [7]

$$[\mathbf{\Pi}(t), \mathbf{\Pi}(t')] \neq 0. \tag{11}$$

Therefore, one must measure the traversal-time in a more physical way. One must demand that if we measure the traversal-time to be t_F, then the particle must actually traverse the distance between x_1 and x_2 in the time given by the traversal-time measurement. For example, one could have a clock which runs when the particle is between x_1 and x_2 given by the Hamiltonian [2] [10]

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\mathbf{Q} \tag{12}$$

where the traversal-time is given by the variable $\mathbf{P}$ conjugate to $\mathbf{Q}$ and the potential V is equal to 1 when $x_1 \leq x \leq x_2$ and 0 everywhere else [2] . In the Heisenberg picture, the equations of motion are

$$\dot{\mathbf{x}} = \mathbf{p}/m, \quad \dot{\mathbf{p}} = -\mathbf{Q}(\delta(\mathbf{x} - x_1) - \delta(\mathbf{x} - x_2)) \tag{13}$$

[1]in our case, where there is no potential barrier, the dwell time, and traversal time are equivalent

[2]The Hamiltonian for this ideal clock is unbounded from below. To remedy this, once could consider a Larmor clock with a bounded Hamiltonian $H_{clock} = \omega \mathbf{J_z}$ [2]. When the particle enters the magnetic field, its spin precesses in the zy-plane. The clock's resolution can be made arbitrarily fine by increasing J_z.

$$\dot{\mathbf{P}} = V(\mathbf{x}), \quad \dot{\mathbf{Q}} = 0. \tag{14}$$

The particle's momentum is disturbed during the measurement.

$$\mathbf{p}' = \sqrt{\mathbf{p}^2 - 2m\mathbf{Q}} \tag{15}$$

where $\mathbf{p}'$ is the particle's momentum during the measurement, and $\mathbf{p}$ is the undisturbed momentum. However if the interaction is weak $Q \ll E_p$, then after a sufficient time, the clock will read the undisturbed traversal-time

$$\begin{aligned} \mathbf{P}(t \to \infty) - \mathbf{P}(0) &\simeq \int_0^\infty V\left(\mathbf{x}(0) - \frac{\mathbf{p}_o t}{m}\right) dt \\ &= \frac{m(x_2 - x_1)}{\mathbf{p}} \end{aligned} \tag{16}$$

If we require an accurate measurement of the traversal-time, then a small dP will result in large values of the coupling Q. If Q is too large, the clock can reflect the particle at x_1 and one will obtain a traversal-time equal to 0. This therefore imposes a restriction on the accuracy with which one can measure the traversal-time. As in Ref. [1] we find that

$$\delta T_F > 1/E_p \tag{17}$$

is required in order to be able to measure the traversal-time, and

$$\delta T_F \gg 1/E_p \tag{18}$$

in order to measure the undisturbed value of the traversal-time.

Let us show that the above conditions are consistent with the uncertainty relations for the position and momentum. If (18) is satisfied, we have $Q \ll E$, and by eq. (15) the momentum during the measurement is

$$\mathbf{p}' \simeq \mathbf{p} - \frac{m}{\mathbf{p}}\mathbf{Q}. \tag{19}$$

Thus during the measurement, the momentum will be uncertain by an amount

$$dp' \simeq \frac{m}{p_o} dQ. \tag{20}$$

In order to know whether the particle entered our detector, we need to be able to distinguish between the case where the pointer is at its initial position $P = 0$, and the case where the particle has gone through the detector $P = t_F = \frac{mL}{p_o}$. We therefore need the condition

$$dP < \frac{mL}{p_o}. \tag{21}$$

Since at best we have $dP = 1/dQ$, we find

$$dp'dx = dp'L > 1. \tag{22}$$

The uncertainty relation (17) only applies to this particular model clock - it might be possible to accurately measure the traversal-time in some clever way. In the following section we will show that this cannot be done, by demonstrating that this uncertainty applies to all measurements of traversal-time.

Finally, we should note that a traversal-time detector could be made by measuring the time-of-arrival to x_1 and the time-of-arrival to x_2. This would require two time-of-arrival clocks each with its own inaccuracy, whereas the model above only has one clock.

IV. MINIMUM UNCERTAINTY FOR TRAVERSAL-TIME

Before proceeding with the argument, we should be clear to distinguish between different types of uncertainties. For an operator $\mathbf{A}$, there exists a kinematic uncertainty which we will denote by $d\mathbf{A}$ given by

$$d\mathbf{A} = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2. \tag{23}$$

This is the uncertainty in the distribution of the observable A. There is also the inherent inaccuracy in the measuring device. This is the relevant uncertainty in equations (1) and (17). It refers to the uncertainty in the initial state of the measuring device's pointer position P, and we will denote it by δA. For our measuring devices we have

$$\delta A = dP_o \tag{24}$$

This uncertainty applies to each individual measurement. Lastly, there is the uncertainty ΔA which applies to the spread in measurements made on the ensemble. Given a set A_M of experiments $i = 1, 2, 3...$ which yield results A_i, we have

$$\Delta A = < A_M^2 > - < A_M >^2 . \tag{25}$$

This uncertainty includes a component due to the kinematic uncertainty of the attribute of the system, and also the inaccuracy of the device. For our measuring device, the kinematic spread in the pointer position at the end of each experiment gives ΔA

$$\Delta A = dP_f \tag{26}$$

The Heisenberg uncertainty relationship $dAdB > 1$ applies to measurements on ensembles. Given an ensemble, we measure **A** on half the ensemble and **B** on the other half. The uncertainty relation also applies to simultaneous measurements. If we measure **A** and **B** simultaneously on each system in the ensemble, then the distributions of **A** and **B** must still satisfy the uncertainty relationship.

Returning now to the traversal-time, we see that it can be interpreted as a simultaneous measurement of position and momentum. We know the particle's momentum p during the time that it was between $x = x_1$ and $x = x_2$ from the classical equations of motion

$$t_F = \frac{mL}{p}. \tag{27}$$

In other words, eigenstates of momentum must have traversal-times given by equation (27). This measurement of momentum is analogous to the measurement described in section II. Instead of measuring the change in position at two specified times t_1 and t_2, we are now measuring the difference in arrival times after specifying two different positions x_1 and x_2. During the measurement, we also know that particle is somewhere between $x = x_1$, and $x = x_2$. ie. we know that $x = \frac{x_1+x_2}{2} \pm L/2$.

The uncertainty relationship also applies to these measured quantities $\Delta x \Delta p > 1$. This essentially means that a detector of size L will disturb the momentum by at least $2/L$,

so that repeated measurements on an ensemble will give $\Delta p > 2/L$. The position of the detector $\mathbf{X}$ computes with the momentum of the particle $\mathbf{p}$ [8] however, we demand that the particle actually travel the distance L. The particle must actually be inside the detector during the measurement. As a result, $\mathbf{X}$ must be coupled to the position $\mathbf{x}$ of the particle and so a measurement of $\mathbf{X}$ is also a measurement of $\mathbf{x}$. This is what we mean by a local interaction.

We can see qualitatively, why we expect (1) to be true. During the measurement of traversal time, the momentum will be disturbed by an amount

$$dp > 2/L. \tag{28}$$

If this disturbance is small, then from (27) we expect this will cause an inaccuracy given by

$$\begin{aligned} \delta T_F &= \frac{mL}{p^2} dp \\ &> 1/E_p \end{aligned} \tag{29}$$

We now proceed with the more rigorous argument. We imagine a traversal-time detector which has an inaccuracy given by δT_F. Measurements can then be carried out on arbitrary ensembles with arbitrary Hamiltonians. We will show that by choosing this ensemble appropriately, the uncertainty relationship $\Delta x \Delta p > 1$ can be violated, unless the traversal-time obeys the relationship given by (1).

We assume that initially, the pointer on our traversal-time detector is given by

$$P_o = \epsilon \tag{30}$$

where ϵ is a small random number which arises because of the initial inaccuracy of the clock. ie. the distribution of ϵ is such that $\langle \epsilon \rangle = 0$ and the clock's initial inaccuracy in pointer position is $dP_o^2 = \langle \epsilon^2 \rangle$. It is important to note that this inaccuracy is fixed as an initial condition before any measurements are made. It is a property of the device, and does not depend on the nature of the ensemble upon which we will be making measurements. For a free Hamiltonian, a measurement of the traversal-time will result in a final pointer position given by

$$P_f = P_o + \frac{mL}{p} \tag{31}$$

where p is the momentum of the particle in the absence of any measuring device. For eigenstates of $\mathbf{p}$ (or states peaked highly in p), we demand that the traversal-time be given by the classically expected value [3] Recall that the kinematic spread in the particle's momentum is given by $d\mathbf{p} = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2$. A measurement of the traversal-time for a particular experiment i can take on the values

$$\begin{aligned} T_i &= P_f \\ &= \frac{mL}{p} + \epsilon \end{aligned} \tag{32}$$

A given measurement T_i will allow us to infer the momentum of the particle p_i during the measurement

$$p_i(T_i) = \frac{mL}{T_i} = \frac{mLp}{mL + p\epsilon}. \tag{33}$$

The average value of any power α of the measured momentum is

$$\langle p_M^\alpha \rangle = \int \left(\frac{mLp}{mL + p\epsilon} \right)^\alpha f(p) g(\epsilon) dp d\epsilon \tag{34}$$

where $f(p)$ gives the distribution of the particle's momentum and $g(\epsilon)$ is the distribution of the fluctuations. We now choose m of the ensemble so that we always have

$$\epsilon p \ll mL. \tag{35}$$

Indeed for the example given in the previous section, for every given ϵ and p, we can increase E_p by choosing a sufficiently large m, and reach this limit. This limit ensures that $\langle p_M \rangle$ never diverges, and simplifies our calculation by allowing us to write

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[3] It is possible to include small deviations from the classical value, by including an additional term in (31). These fluctuations need to average to zero in order to satisfy the correspondence principle. For small fluctuations, the following discussion is not altered.

$$\langle p_M^\alpha \rangle \simeq \int \left(p - \frac{\epsilon p^2}{mL} \right)^\alpha f(p) g(\epsilon) dp d\epsilon \tag{36}$$

For $\alpha = 1$ we find

$$\begin{aligned} \langle p_M \rangle &\simeq \langle p \rangle - \frac{\langle \epsilon \rangle \langle p^2 \rangle}{mL} \\ &= \langle p \rangle. \end{aligned} \tag{37}$$

For $\alpha = 2$ we find

$$\langle p_M^2 \rangle \simeq \int \left(p^2 - 2\frac{\epsilon p^3}{mL} + (\frac{\epsilon p^2}{mL})^2 \right) f(p) g(\epsilon) dp d\epsilon \tag{38}$$

$$= \langle p^2 \rangle + \frac{\langle p^4 \rangle \langle \epsilon^2 \rangle}{(mL)^2}. \tag{39}$$

This gives us

$$\begin{aligned} \Delta p^2 &= \langle p_M^2 \rangle - \langle p_M \rangle^2 \\ &= \frac{\langle p^4 \rangle \delta T_F^2}{(mL)^2} + dp^2 \end{aligned} \tag{40}$$

Since

$$(dE)^2 = \frac{\langle p^4 \rangle}{4m^2} - \langle E \rangle^2 \tag{41}$$

we find

$$\Delta p^2 = (\frac{2\delta T_F}{L})^2 (dE^2 + \langle E \rangle^2) + dp^2. \tag{42}$$

Finally, we arrive at the relation

$$(\Delta x \Delta p)^2 = \delta T_F^2 (\langle E \rangle^2 + dE^2) + \frac{L^2}{4} dp^2. \tag{43}$$

The uncertainty relation

$$\Delta x \Delta p > 1 \tag{44}$$

then implies

$$\delta T_F^2 > \frac{1 - \frac{1}{4}L^2 dp^2}{\langle E\rangle^2 + dE^2}. \tag{45}$$

Now we note that we can arrange our experiment with Ldp arbitrarily small, by choosing dp of the ensemble arbitrarily small. As a result, in order to ensure that Heisenberg's uncertainty relation is never violated, we must have

$$\delta T_F > \frac{1}{\sqrt{\langle E\rangle^2 + dE^2}}. \tag{46}$$

The condition (35) and (46) imply that we have $dE \ll E$, so we can write

$$\delta T_F > \frac{1}{\langle E\rangle}. \tag{47}$$

It is interesting to note that since the momentum operator commutes with the free Hamiltonian, the restriction on traversal-time measurements only comes from the dynamical considerations given above.

V. CONCLUSION

We have seen that the measurement of the traversal-time given two positions cannot be made arbitrarily accurate. This strongly suggests that the limitation on measurements of arrival times is a general rule and not just an artifact of the types of models considered so far. Operators for both these quantities are singular or don't seem to correspond to physical (continuous) processes. The case of traversal-time is different from time-of-arrival in that the semi-bounded spectrum of the Hamiltonian does not seem to play an important role in the restriction on measurement accuracy. The accuracy restriction on traversal-time is particularly important for experiments on barrier tunnelling time. One usually uses a physical clock to measure the time it takes for a particle to travel from one location to another, with a barrier situated somewhere between the two locations [9] [10]. These measurements need to be inherently inaccurate, because if one tries to measure the tunnelling time too accurately, the particle will not be able to tunnel. Our result concerning traversal-time indicates that the barrier tunnelling time also cannot be precisely defined.

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